Credibility Theory

Ana-Iulia DOMNISAN

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Introduction

- Origins.
- Definition.
- Example.
- Approaches.

Definition

- mathematical tool
- deals with the randomness of data
- helps predict future events or costs

The First Formula

- basic formula for credibility weighted estimates

$$\textbf{Estimate} = \mathrm{Z} \times [\textit{Observation}] + (1 - Z) \times [\textit{Other Information}]$$

- Z is the credibility assigned to the observation
- 1−Z is referred to as the complement of credibility
- $0 \leq Z \leq 1$

A Simple Example

- large population of drivers observed over a 5-year period.
- **average** driver has an annual frequency of 0.20 accidents per year.
- **random** driver has an annual frequency of 0.60 accidents per year.
- Question: What is the estimate of the expected future frequency rate for this driver?

Solution

- Dilemma! 0.20, 0.60 or something in between?
- Best Solution: this driver's

  Expected Future Accident Frequency $= Z \times 0.60 + (1 - Z) \times 0.20$

- Attention: anti-selection effect!

Approaches

- **Classical Credibility:**
  – limited fluctuation credibility model.
  – homogenous risk classes.

- **Bühlmann Credibility:**
  – least squares credibility model.

- **Bayesian Analysis:**
  – formulas match those of Bühlmann credibility estimation.
  – linear weightening of current and prior information
  with weights Z and (1 - Z) where Z is the Bühlmann credibility.

The Model

- insurance company
- I insured risks, numbered $i = 1, 2, \ldots, I$
- in a well-defined insurance period, the risk i produces:
  - a number of claims N_i,
  - with claim sizes $Y_i^{(\nu)}$, where $\nu = 1, 2, \ldots, N_i$,
  - which together give the aggregate claim amount

$$X_i = \sum_{\nu=1}^{N_i} Y_i^{(\nu)}$$

- gross premium
- premium volume
- task: determine the pure risk premium $P_i = E[X_i]$.

The Individual Risk

- black box
- X_j ($j = 1, 2, \ldots, n$) denotes the claim amount during the time period j
- previous periods: $\mathbb{X} = (X_1, \ldots, X_n)'$
- present period: X_{n+1}
- standard assumptions regarding the distribution function $F(x)$ of the random variables X_j:
  – stationarity,
  – (conditional) independence.
- parameterisation: ϑ = "risk profile", element of the set Θ.

From the Fictional to the True Premium

- The Two-urn Model

- **The individual premium:**

$$P^{ind} = \mu(\Theta) = E[\, X_{n+1} | \mathbb{X} \,]$$

- **The collective premium:**

$$P^{coll} = \mu_0 = \int_{\Theta} \mu(\vartheta)\, dU(\vartheta) = E[\, X_{n+1} \,]$$

- **The Bayes premium:**

$$P^{Bayes} = \overline{\mu(\Theta)} = E[\, \mu(\Theta) | \mathbb{X} \,]$$

Statistical Decision Theory

- observation vector $\mathbb{X} = (X_1, X_2, \ldots, X_n)'$
- distribution function $F_\vartheta(x) = P_\vartheta[\mathbb{X} \leq \mathbf{x}]$
- GOAL: the value of a specific functional $g(\vartheta)$ of the parameter ϑ
- $T(\mathbb{X})$ which:
  - depends only on $\mathbb{X}$,
  - will estimate $g(\vartheta)$ "as well as possible".

Formulation of the Problem

- $\vartheta \in \Theta$: set of parameters, which contains the true value of ϑ
- $T \in D$: set of functions to which the estimator function must belong.
- $D = \{g(\vartheta) : \vartheta \in \theta\}$
- *loss function*: $L(\vartheta, T(\mathbf{x}))$
- *risk function*:

$$R_T(\vartheta) := E_\vartheta[L(\vartheta, T)] = \int_{\mathbb{R}^n} L(\vartheta, T(\mathbf{x}))dF(\mathbf{x}) .$$

Bayes Risk, Bayes Estimator

- $U(\vartheta)$ **a priori** distribution for Θ.
- **Bayes risk**:

$$R(T) := \int_{\Theta} R_T(\vartheta) dU(\vartheta).$$

- **Bayes Estimator** $\overline{T}$:

$$\overline{T} := \underbrace{arg\ min}_{T \in D} R(T).$$

The Quadratic Loss

- correct individual premium $\mu(\vartheta)$
- $\Theta \; =$ the set of individual risk profiles ϑ.
- quadratic loss function

$$L(\vartheta, T(\mathbf{x})) = (\mu(\vartheta) - T(\mathbf{x}))^2$$

- the Bayes estimator with respect to the quadratic loss function is given by

$$\overline{\mu(\Theta)} = E\left[\mu(\Theta)|\mathbb{X}\right]$$

The Bayes Premium in Three Special Cases

- *1.* **The Poisson–Gamma Case**
- *2.* **The Binomial–Beta Case**
- *3.* **The Normal–Normal Case**

1. The Poisson–Gamma Case

- **Motivation**: F.Bichsel's Problem

  – 1960's in Switzerland,

  – Bonus-Malus System was created,

  – premium level based on horsepower of the car,

  – Task: construct a better risk premium adjusted to the individuals risk profile.

  – differences in individual numbers of claim.

Mathematical Modelling

- N_j claims during year j
- corresponding aggregate claim amount X_j.
- implicit assumption of Bichsel:
  - given: individual risk profile ϑ,
  - $E[X_j|\Theta = \vartheta] \; = \; C \; E[N_j|\Theta = \vartheta]$ holds the aggregate claim amount,
  - C - constant depending on the horsepower of the car,
  - $E[N_j|\Theta = \vartheta]$ depends only on the driver.

Model Assumptions

- *PG1:* Conditionally, given $\Theta = \vartheta$, the N_j's $(j = 1,\ 2, \ldots, n)$ are independent and Poisson distributed with Poisson parameter ϑ:

$$P(N_j = k|\Theta = \vartheta) \;=\; e^{-\vartheta}\ \frac{\vartheta^k}{k!}.$$

- *PG2:* Θ has a Gamma distribution with shape parameter γ and scale parameter β, the structural function has density:

$$u(\vartheta) \;=\; \frac{\beta^\gamma}{\Gamma(\gamma)}\ \vartheta^{\gamma-1} e^{-\beta\vartheta}, \quad \vartheta \geq 0.$$

The Claim Frequencies

- **The individual claim frequency:**

$$F^{ind} = E[N_{n+1}|\Theta] = \Theta.$$

- **The collective claim frequency:**

$$F^{coll} = E[\Theta] = \frac{\gamma}{\beta}.$$

The Bayes Claim Frequency:

- **Claim Frequency:**

$$F^{Bayes} = \frac{\gamma + N_\bullet}{\beta + n} = \alpha \overline{N} + (1-\alpha) \frac{\gamma}{\beta}$$

$$\text{where } \alpha = \frac{n}{n+\beta}, \quad \overline{N} = \frac{1}{n} \sum_{j=1}^{n} N_j .$$

- The *quadratic loss* of F^{Bayes} is:

$$\mathrm{E}\left[\left(\mathrm{F}^{Bayes} - \Theta\right)^2\right] = (1 - \alpha)\, E\left[\left(F^{coll} - \Theta\right)^2\right]$$

$$= \alpha\, E\left[\left(\overline{N} - \Theta\right)^2\right].$$

Remarks

- The quantities P^{ind}, P^{coll} and P^{Bayes} are obtained by multiplication with C.
- The Bayes premium CF^{Bayes} is a linear function of the observations (*credibility premium*) .
- F^{Bayes} is an average of:
  - $\overline{N}$ = observed individual claim frequency and
  - $E[\Theta] = \frac{\gamma}{\beta}$ = a priori expected claim frequency.

Remarks

- $\alpha = \frac{n}{n+\beta}$ – credibility weight.
- great number of obervation years n leads to a large α .
- if $\beta = \frac{E\left[\Theta\right]}{Var\left[\Theta\right]}$ large then α small.
- Explanation of the quatradic loss:
  - the quadratic loss of $F^{Bayes} = (1 - \alpha)\cdot$ quadratic loss of F^{coll}
  - the quadratic loss of $F^{Bayes} = \alpha\cdot$ quadratic loss of $\overline{N}$

Estimators

- F^{coll} is the estimator based only on the a priori knowledge from the collective, neglecting the individual claim experience.
- $\overline{N}$ is the estimator based only on the individual claims experience, neglecting the apriori knowledge.
- F^{Bayes} combines both sources of information.

2. The Binomial–Beta Case

- **Motivation:**

  – group life/accident insurance,

  – interesting: number of disability cases or disability frequency for a certain group.

- A few assumptions to simplify matters:

  – each member of the group has the same probability of disablement,

  – disabilities occur independently

  – disabled person leaves group.

Random Variables

- N_j = number of new disabilities occuring in the group in the year $j = 1, 2, \ldots$
- V_j = number of (not disabled) members in the group at the beginning of year $j = 1, 2, \ldots$
- $X_j = \frac{N_j}{V_j}$ observed disablement frequency in year $j = 1, 2, \ldots$
- of interest is:

$$X_{n+1} = \frac{N_{n+1}}{V_{n+1}}$$

Model Assumptions

- *BB1:* Conditionally, given $\Theta = \vartheta$, N_j $(j = 1,\ 2, \ldots, n)$ are independent and binomial distributed with :

$$P\left[N_j = k|\Theta\right] = \binom{V_j}{k}\vartheta^k\ (1\ -\ \vartheta)^{V_j-k}.$$

- *BB2:* Θ has a Beta(a, b) distribution with $a, b >\ 0$, equivalently, the structural function has density:

$$u(\vartheta)\ =\ \frac{1}{B(a,b)}\ \vartheta^{a-1}\ (1-\vartheta)^{b-1},\quad 0 \leq \vartheta \leq 1.$$

$$\text{where } B(a, b)\ =\ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

The Claim Frequencies

- **The individual claim frequency:**

$$F^{ind} = E[X_{n+1}|\Theta] = \Theta.$$

- **The collective claim frequency:**

$$F^{coll} = E[\Theta] = \frac{a}{a+b}.$$

Bayes Claim Frequency:

- **Claim Frequency:**

$$F^{Bayes} = \frac{a+N_\bullet}{a+b+N_\bullet} = \alpha \overline{N} + (1-\alpha)\frac{a}{a+b}$$

$$\text{where } \overline{N} = \frac{N_\bullet}{V_\bullet}, \quad \alpha = \frac{V_\bullet}{a+b+V_\bullet}.$$

- The *quadratic loss* of F^{Bayes} is:

$$\mathrm{E}\left[\left(\mathrm{F}^{Bayes} - \Theta\right)^2\right] = (1 - \alpha)\ E\left[\left(F^{coll} - \Theta\right)^2\right]$$

$$= \alpha\ E\left[\left(\overline{N} - \Theta\right)^2\right].$$

The Normal–Normal Case - Remarks

- no practical motivation
- insurance data is mostly not normally distributed
- sometimes useful for large portfolios

- consider an individual risk
- observation vector: $\mathbb{X} = (X_1, \ldots, X_n)'$
- X_j aggregate claim amount in the $j - th$ year

Model Assumptions

- *NN1:* Conditionally, given $\Theta = \vartheta$, the X_j's $(j = 1,\ 2, \ldots, n)$ are independent and normally distributed with :

$$X_j \ \sim \ \mathcal{N}\left(\vartheta, \sigma^2\right).$$

- *NN2:* $\Theta \ \sim \ \mathcal{N}\left(\mu, \tau^2\right)$,
  and the structural function has the density:

$$u(\vartheta) \ = \ \frac{1}{\sqrt{2\pi}} \ e^{-\frac{1}{2}\left(\frac{\vartheta \ - \ \mu}{\tau}\right)^2} \quad .$$

The Premiums

- **The individual Premium:**

$$P^{ind} = E[X_{n+1}|\Theta] = \Theta.$$

- **The collective claim frequency:**

$$P^{coll} = E[\Theta] = \mu.$$

Bayes Premium:

- **Premium:**

$$P^{Bayes} = \frac{\tau^2\mu+\sigma^2 X_\bullet}{\tau^2+n\sigma^2} = \alpha\overline{X} + (1-\alpha)E\left[\Theta\right]$$

$$\text{where } \overline{X} = \frac{1}{n}X_\bullet, \quad \alpha = \frac{n}{n+\frac{\sigma^2}{\tau^2}}.$$

- The *quadratic loss* of P^{Bayes} is:

$$\mathrm{E}\left[\left(\mathrm{P}^{Bayes} - \Theta\right)^2\right] = (1 - \alpha)\ E\left[\left(P^{coll} - \Theta\right)^2\right]$$

$$= \alpha\ E\left[\left(\overline{X} - \Theta\right)^2\right].$$

Common Features

- the Bayes Premium is a linear function of the observations $\Longrightarrow$ a credibility premium
- P^{Bayes} can be expressed as a weighted mean:

$$P^{Bayes} = \alpha \overline{X} + (1 - \alpha) P^{coll}.$$

- the weight α is given by:

$$\alpha = \frac{n}{n+\kappa}, \text{ where } \kappa \text{ is an appropiate constant.}$$

- the quadratic loss of the Bayes premium is:

$$\mathrm{E}\left[\left(\mathrm{P}^{Bayes} - \Theta\right)^2\right] = (1 - \alpha) E\left[\left(P^{coll} - \Theta\right)^2\right] = \alpha E\left[\left(\overline{X} - \Theta\right)^2\right].$$

Thank You for Your Attention!